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Kernel solutions of the Kostant operator on eight-dimensional quotient spaces

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Abstract: After introducing the generators and irreducible representations of the su(5) and so(6) Lie algebras in terms of the Schwinger's oscillators, the general kernel solutions of the Kostant operators on eight-dimensional quotient spaces su(5)/su(4) × u(1) and so(6)/so(4) × so(2) are derived in terms of the diagonal subalgebras su(4) × u(1) and so(4) × so(2), respectively.

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Contents

1. Introduction

The Dirac operator plays a significant role in quantum field theories. Its natural generalization with a cubic term arose from Kazama and Suzuki's attempt to create a realistic string model [1]. Their cubic Dirac operator appeared in the string model as a supercurrent of a superconformal algebra. Ten years later, this kind of operator was discovered again by Kostant [2]. He understood already that an Euler number multiplet from an equal rank embedding of reductive Lie algebras [3] is nothing more than kernel solutions of the cubic Dirac operator. It is also an accident that the lowest lines of the Euler number multiplets for the 4-, 8-, and 16-dimensional coset spaces match with the known supersymmetric multiplets [4].

Although, the Euler number multiplets are easily derived by the GKRS index formula [3],

$$S^+ \otimes V_\Lambda - S^- \otimes V_\Lambda = \sum_{c\in C} \mathrm{sgn}(c) U_{c\bullet\Lambda},$$

they are not helpful for the formulation of any physical theory. In [5], Brink, Ramond and Xiong used an algebraic method to determine the general kernel solutions or the Euler number multiplets of the Kostant operators on the cosets SU(3)/SU(2) × U(1) and F_4/SO(9). By realizing the gamma matrices as dynamical variables satisfying Grassmann algebras, the Euler number triplets for SU(3)/SU(2)×U(1) and F_4/SO(9) were then written as chiral superfields. A free action in the light-cone frame for both cosets was also formulated.

The intention of this paper is to determine the general kernel solutions of the Kostant operators on the 8-dimensional quotients su(5)/su(4) × u(1) and so(6)/so(4) × so(2) by a quantum mechanical method. We will briefly present how to construct the generators of su(5) and so(6) Lie algebras and their irreducible representations (irreps). Only parts that

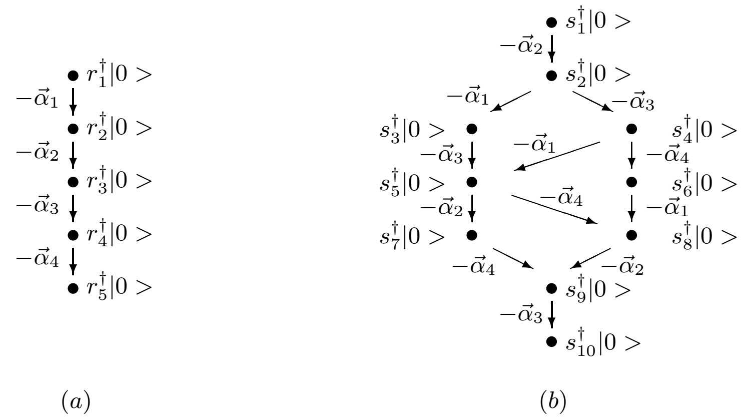

Figure 1: The su(5) weight diagrams (a) of a 5-dimensional and (b) of a 10-dimensional irreps.

are used in constructing the Kostant operators will be mentioned. Then, the general kernel solutions will be determined. Their extension to the case of a non-compact Lie algebra was originated in 1999 by Ramond from his curiosity to know the Euler number multiplets. Some comments about them are made in the last section.

2. Kostant operator of the quotient su(5)/su(4) × u(1) and its kernel solutions

2.1 The Schwinger's oscillator realization of the su(5) Lie algebra

To construct the su(5) generators that satisfy Chevalley-Serre relations [6], we introduce four types of Schwinger's oscillators $r_i,\ \bar{r}_i,\ s_j,\ \bar{s}_j$, where $i = 1$ to 5 and $j = 1$ to 10, including their adjoints [7]. Action of the raising oscillators $r_i^\dagger$ and $s_j^\dagger$ on the vacuum state in correspondence to the su(5) irreps **5** and **10** is shown in figure 1a and 1b, respectively. By reversing all arrows in figure 1a and 1b, and replacing $r_i^\dagger$ and $s_j^\dagger$ with $\bar{r}_i^\dagger$ and $\bar{s}_j^\dagger$, they become the weight diagrams of the $\overline{\mathbf{5}}$ and $\overline{\mathbf{10}}$ irreps. Although, the **10** and $\overline{\mathbf{10}}$ irreps are not fundamental and can be obtained from anti-symmetrization of **5** and $\overline{\mathbf{5}}$ irreps, respectively, it will be seen later that introducing the oscillators $s_j,\ \bar{s}_j$ and their adjoints is a convenient way in determining the general kernel solutions.

From the **5**, $\overline{\mathbf{5}}$, **10** and $\overline{\mathbf{10}}$ weight diagrams, all positive root generators are

$$\begin{aligned}
T_1^+ &= r_1^\dagger r_2 + s_2^\dagger s_3 + s_4^\dagger s_5 + s_6^\dagger s_8 + (r, s \to \bar{r}, \bar{s})^\dagger,\\
T_2^+ &= r_2^\dagger r_3 + s_1^\dagger s_2 + s_5^\dagger s_7 + s_8^\dagger s_9 + (r, s \to \bar{r}, \bar{s})^\dagger,\\
T_3^+ &= r_1^\dagger r_3 - s_1^\dagger s_3 + s_4^\dagger s_7 + s_6^\dagger s_9 + (r, s \to \bar{r}, \bar{s})^\dagger,\\
T_4^+ &= r_3^\dagger r_4 + s_2^\dagger s_4 + s_3^\dagger s_5 + s_9^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger,\\
T_5^+ &= r_2^\dagger r_4 + s_1^\dagger s_4 - s_3^\dagger s_7 + s_8^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger,\\
T_6^+ &= r_1^\dagger r_4 - s_1^\dagger s_5 - s_2^\dagger s_7 + s_6^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger,\\
T_7^+ &= r_4^\dagger r_5 + s_4^\dagger s_6 + s_5^\dagger s_8 + s_7^\dagger s_9 + (r, s \to \bar{r}, \bar{s})^\dagger,\\
T_8^+ &= r_3^\dagger r_5 + s_2^\dagger s_6 + s_3^\dagger s_8 - s_7^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger,
\end{aligned}$$

$$T_9^+ = r_2^\dagger r_5 + s_1^\dagger s_6 - s_3^\dagger s_9 - s_5^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger,$$
$$T_{10}^+ = r_1^\dagger r_5 - s_1^\dagger s_8 - s_2^\dagger s_9 - s_4^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger, \quad (2.1)$$

and all negative ones are

$$T_A^- = \left(T_A^+\right)^\dagger, \qquad A = 1,\ 2,\ 3,\ \ldots,\ 10. \quad (2.2)$$

The Cartan subalgebra generators in the Dynkin basis,

$$\vec{H} \equiv H_1\hat{\omega}_1 + H_2\hat{\omega}_2 + H_3\hat{\omega}_3 + H_4\hat{\omega}_4 = (H_1, H_2, H_3, H_4), \quad (2.3)$$

are obtained from the following commutators:

$$\begin{aligned}
H_1 &\equiv [T_1^+, T_1^-] \\
&= N_1^{(r)} + N_2^{(s)} + N_4^{(s)} + N_6^{(s)} - N_2^{(r)} - N_3^{(s)} - N_5^{(s)} - N_8^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_2 &\equiv [T_2^+, T_2^-] \\
&= N_2^{(r)} + N_1^{(s)} + N_5^{(s)} + N_8^{(s)} - N_3^{(r)} - N_2^{(s)} - N_7^{(s)} - N_9^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_3 &\equiv [T_4^+, T_4^-] \\
&= N_3^{(r)} + N_2^{(s)} + N_3^{(s)} + N_9^{(s)} - N_4^{(r)} - N_4^{(s)} - N_5^{(s)} - N_{10}^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_4 &\equiv [T_7^+, T_7^-] \\
&= N_4^{(r)} + N_4^{(s)} + N_5^{(s)} + N_7^{(s)} - N_5^{(r)} - N_6^{(s)} - N_8^{(s)} - N_9^{(s)} - (r, s \to \bar{r}, \bar{s}), \quad (2.4)
\end{aligned}$$

where $N_i^{r,s,\bar{r},\bar{s}}$ are the number operators. They are related to the Cartan generators in the orthonormal basis,

$$\vec{h} \equiv h_1\hat{e}_1 + h_2\hat{e}_2 + h_3\hat{e}_3 + h_4\hat{e}_4 + h_5\hat{e}_5 = [h_1, h_2, h_3, h_4, h_5], \quad (2.5)$$

as follows:

$$H_1 = h_1 - h_2,\ H_2 = h_2 - h_3,\ H_3 = h_3 - h_4,\ H_4 = h_4 - h_5. \quad (2.6)$$

An su(5) irrep represented by its highest weight Λ in its vector space V_Λ can be generated by action of the raising oscillators, $r_1^\dagger,\ \bar{r}_5^\dagger,\ s_1^\dagger,\ \bar{s}_{10}^\dagger$, on the vacuum state

$$\Lambda = (r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_{10}^\dagger)^{a_3}(\bar{r}_5^\dagger)^{a_4}|0>, \quad (2.7)$$

where $a_{1,2,3,4}$ are non-negative integers, called the Dynkin labels. The action of the Cartan generators in the Dynkin basis on the highest weight gives their eigenvalues as follow:

$$\vec{H}\Lambda = (a_1, a_2, a_3, a_4)\Lambda, \quad (2.8)$$

and in the orthonormal basis as follow:

$$\vec{h}\Lambda = [b_1, b_2, b_3, b_4, b_5]\Lambda, \quad (2.9)$$

where

$$b_1 = \frac{1}{5}(4a_1 + 3a_2 + 2a_3 + a_4),$$

$$
\begin{aligned}
b_2 &= \frac{1}{5}(-a_1 + 3a_2 + 2a_3 + a_4), \\
b_3 &= \frac{1}{5}(-a_1 - 2a_2 + 2a_3 + a_4), \\
b_4 &= \frac{1}{5}(-a_1 - 2a_2 - 3a_3 + a_4), \\
b_5 &= \frac{1}{5}(-a_1 - 2a_2 - 3a_3 - 4a_4).
\end{aligned} \tag{2.10}
$$

Note that $b_1 + b_2 + b_3 + b_4 + b_5 = 0$ is due to the basis constraint.

Inside the su(5) generators, the generators $T^{\pm}_{1,2,\ldots,6}$ and $H_{1,2,3}$ form the su(4) Lie subalgebra and the generator h_5 is the generator of u(1) subalgebra. The other generators $T^{\pm}_{7,8,9,10}$ lie outside the subalgebra $\mathrm{su}(4)\times\mathrm{u}(1)$ and they are used to construct the Kostant operator of the quotient $\mathrm{su}(5)/\mathrm{su}(4)\times\mathrm{u}(1)$.

2.2 Kernel solutions of the Kostant operator

To construct the Kostant operator on the 8-dimensional quotient space, the following 16×16 gamma matrices are needed:

$$
\begin{aligned}
\Gamma_1 &= \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma_1, & \Gamma_5 &= \sigma_1\otimes\sigma_1\otimes\sigma_3\otimes\mathbb{1}, \\
\Gamma_2 &= \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma_2, & \Gamma_6 &= \sigma_1\otimes\sigma_2\otimes\mathbb{1}\otimes\mathbb{1}, \\
\Gamma_3 &= \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma_3, & \Gamma_7 &= \sigma_1\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1}, \\
\Gamma_4 &= \sigma_1\otimes\sigma_1\otimes\sigma_2\otimes\mathbb{1}, & \Gamma_8 &= \sigma_2\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1},
\end{aligned}
$$

where $\sigma_{1,2,3}$ are the Pauli matrices and $\mathbb{1}$ is a 2×2 identity matrix. These gamma matrices satisfy Clifford algebra

$$
\{\Gamma_a, \Gamma_b\} = 2\delta_{a,b}\,(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1})\,. \tag{2.11}
$$

To associate with the generators of the quotient $\mathrm{su}(5)/\mathrm{su}(4)\times\mathrm{u}(1)$, the gamma matrices are complexified as follows:

$$
\begin{aligned}
\gamma_7^{\pm} &= \frac{1}{2}(\Gamma_1 \pm i\Gamma_2) = \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma^{\pm}, \\
\gamma_8^{\pm} &= \frac{1}{2}(\Gamma_3 \pm i\Gamma_4) = \sigma_1\otimes\sigma_1\otimes\left[\sigma^{+}\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1}) + \sigma^{-}\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1})\right], \\
\gamma_9^{\pm} &= \frac{1}{2}(\Gamma_5 \pm i\Gamma_6) = \sigma_1\otimes\left[\sigma^{+}\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1}) + \sigma^{-}\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1})\right]\otimes\mathbb{1}, \\
\gamma_{10}^{\pm} &= \frac{1}{2}(\Gamma_7 \pm i\Gamma_8) = \left[\sigma^{+}\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1}) + \sigma^{-}\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1})\right]\otimes\mathbb{1}\otimes\mathbb{1}.
\end{aligned} \tag{2.12}
$$

Under these complexification, the positive spinor states of so(8) are $|+\pm\pm\pm>$ and the negative ones $|-\pm\pm\pm>$.

From the commutators of the generators of the quotient,

$$
\begin{aligned}
[T_7^+, T_7^-] &= h_4 - h_5, & [T_8^+, T_8^-] &= h_3 - h_5, \\
[T_9^+, T_9^-] &= h_2 - h_5, & [T_{10}^+, T_{10}^-] &= h_1 - h_5,
\end{aligned} \tag{2.13}
$$

the generators $T^{\pm}_{a=7,8,9,10}$ are not generated. The structure constants of these transformations are zero. Hence, there are no cubic terms, which are composed of a product of three gamma matrices associated with the structure constants. The Kostant operator of the quotient su(5)/su(4) × u(1) is just

$$\not{K} = \sum_{a=7}^{10} (\gamma_a^+ T_a^- + \gamma_a^- T_a^+). \tag{2.14}$$

This Kostant operator acts on a tensor-product space of the so(8) spinor representations and the su(5) irrep

$$\psi_\Lambda^\pm \equiv | \pm \pm \pm \pm > V_\Lambda, \tag{2.15}$$

and there exist kernel solutions such that

$$\not{K} \psi_{\lambda_i}^\pm = 0, \tag{2.16}$$

where λ_i is a weight in the vector space V_Λ. Equation (2.16) can be decomposed into sixteen, independent equations as follows:

$$\begin{aligned}
(T_7^+ + T_8^+ + T_9^+ + T_{10}^+)\psi_{\lambda_1}^+ &= 0, \\
(T_7^- - T_8^- + T_9^+ + T_{10}^+)\psi_{\lambda_2}^+ &= 0, \\
(T_7^+ + T_8^- - T_9^- + T_{10}^+)\psi_{\lambda_3}^+ &= 0, \\
(T_7^- - T_8^+ - T_9^- + T_{10}^+)\psi_{\lambda_4}^+ &= 0, \\
(T_7^+ + T_8^+ + T_9^- - T_{10}^-)\psi_{\lambda_5}^+ &= 0, \\
(T_7^- - T_8^- + T_9^- - T_{10}^-)\psi_{\lambda_6}^+ &= 0, \\
(T_7^+ + T_8^- - T_9^+ - T_{10}^-)\psi_{\lambda_7}^+ &= 0, \\
(T_7^- - T_8^+ - T_9^+ - T_{10}^-)\psi_{\lambda_8}^+ &= 0, \\
(T_7^+ + T_8^+ + T_9^+ + T_{10}^-)\psi_{\lambda_1'}^- &= 0, \\
(T_7^- - T_8^- + T_9^+ + T_{10}^-)\psi_{\lambda_2'}^- &= 0, \\
(T_7^+ + T_8^- - T_9^- + T_{10}^-)\psi_{\lambda_3'}^- &= 0, \\
(T_7^- - T_8^+ - T_9^- + T_{10}^-)\psi_{\lambda_4'}^- &= 0, \\
(T_7^+ + T_8^+ + T_9^- - T_{10}^+)\psi_{\lambda_5'}^- &= 0, \\
(T_7^- - T_8^- + T_9^- - T_{10}^+)\psi_{\lambda_6'}^- &= 0, \\
(T_7^+ + T_8^- - T_9^+ - T_{10}^+)\psi_{\lambda_7'}^- &= 0, \\
(T_7^- - T_8^+ - T_9^+ - T_{10}^+)\psi_{\lambda_8'}^- &= 0.
\end{aligned} \tag{2.17}$$

One of the possible kernel solutions in the positive spinor space is as follows:

$$\begin{aligned}
\psi_{\lambda_1}^+ &= | + + + + > (r_1^\dagger)^{a_1} (s_1^\dagger)^{a_2} (\bar{s}_{10}^\dagger)^{a_3} (\bar{r}_5^\dagger)^{a_4} |0>, \\
\psi_{\lambda_2}^+ &= | + + + - > (r_1^\dagger)^{a_1} (s_1^\dagger)^{a_2} (\bar{s}_7^\dagger)^{a_3} (\bar{r}_4^\dagger)^{a_4} |0>, \\
\psi_{\lambda_3}^+ &= | + + - + > (r_1^\dagger)^{a_1} (s_4^\dagger)^{a_2} (\bar{s}_3^\dagger)^{a_3} (\bar{r}_3^\dagger)^{a_4} |0>,
\end{aligned}$$

$$
\begin{aligned}
\psi^+_{\lambda_4} &= |++--> (r_1^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_5^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_5} &= |+-++> (r_4^\dagger)^{a_1}(s_7^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_6} &= |+-+-> (r_5^\dagger)^{a_1}(s_6^\dagger)^{a_2}(\bar{s}_7^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_7} &= |+--+> (r_2^\dagger)^{a_1}(s_5^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_8} &= |+---> (r_3^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,
\end{aligned} \tag{2.18}
$$

and in the negative spinor space as follows:

$$
\begin{aligned}
\psi^-_{\lambda'_1} &= |-+++> (r_4^\dagger)^{a_1}(s_7^\dagger)^{a_2}(\bar{s}_6^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_2} &= |-++-> (r_2^\dagger)^{a_1}(s_8^\dagger)^{a_2}(\bar{s}_7^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_3} &= |-+-+> (r_4^\dagger)^{a_1}(s_{10}^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_4} &= |-+--> (r_3^\dagger)^{a_1}(s_9^\dagger)^{a_2}(\bar{s}_5^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_5} &= |--++> (r_4^\dagger)^{a_1}(s_7^\dagger)^{a_2}(\bar{s}_8^\dagger)^{a_3}(\bar{r}_2^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_6} &= |--+-> (r_1^\dagger)^{a_1}(s_6^\dagger)^{a_2}(\bar{s}_7^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_7} &= |---+> (r_4^\dagger)^{a_1}(s_5^\dagger)^{a_2}(\bar{s}_9^\dagger)^{a_3}(\bar{r}_3^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_8} &= |----> (r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_{10}^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>.
\end{aligned} \tag{2.19}
$$

To get the kernel solutions in terms of su(4) $\times$ u(1), it needs to act on them by the Cartan subalgebra generators, which in the Dynkin basis are

$$
\begin{aligned}
D_1 &= h_1 - h_2 + \frac{1}{2}\left(f^{10}_{+-1}[\gamma^+_{10},\gamma^-_{10}] - f^{9}_{+-2}[\gamma^+_9,\gamma^-_9]\right)\\
&= H_1 + \frac{1}{2}\left(\sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1}\right),\\
D_2 &= h_2 - h_3 + \frac{1}{2}\left(f^{9}_{+-2}[\gamma^+_9,\gamma^-_9] - f^{8}_{+-3}[\gamma^+_8,\gamma^-_8]\right)\\
&= H_2 + \frac{1}{2}\left(\mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1} - \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3\right),\\
D_3 &= h_3 - h_4 + \frac{1}{2}\left(f^{8}_{+-3}[\gamma^+_8,\gamma^-_8] - f^{7}_{+-4}[\gamma^+_7,\gamma^-_7]\right)\\
&= H_3 + \frac{1}{2}\left(\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3 - \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\right),\\
D_4 &= \frac{1}{2}h_5 + \frac{1}{4}\left(f^{7}_{+-5}[\gamma^+_7,\gamma^-_7] + f^{8}_{+-5}[\gamma^+_8,\gamma^-_8] + f^{9}_{+-5}[\gamma^+_9,\gamma^-_9] + f^{10}_{+-5}[\gamma^+_{10},\gamma^-_{10}]\right)\\
&= \frac{1}{2}h_5 - \frac{1}{4}(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3 + \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3 + \mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1}\\
&\qquad + \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1}).
\end{aligned} \tag{2.20}
$$

The structure constants in (2.20) are read directly from (2.13). The generators D_1, D_2 and D_3 are the Cartan generators of su(4) and the generator D_4 is the Cartan generator of u(1). When the Cartan generators act on the kernel solutions, they give

$$
(D_1, D_2, D_3; D_4)\psi^+_{\lambda_1} = (a_1, a_2, a_3; (b_5-2)/2)\psi^+_{\lambda_1},
$$

$$
\begin{aligned}
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_2} &= (a_1,a_2+a_3+1,a_4;b_3/2)\psi^+_{\lambda_2},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_3} &= (a_1+a_2+a_3+1,a_4,-a_2-a_3-a_4-1;b_3/2)\psi^+_{\lambda_3},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_4} &= (a_1+a_2+a_3+1,-a_2-a_3-1,a_2+a_3+a_4+1;b_3/2)\psi^+_{\lambda_4},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_5} &= (-a_4,-a_2-a_3-1,-a_1;b_3/2)\psi^+_{\lambda_5},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_6} &= (a_2,a_3,a_4;(b_1+2)/2)\psi^+_{\lambda_6},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_7} &= (-a_1-a_2-a_3-a_4-1,a_1+a_2+a_3+1,-a_2-a_3-1;b_3/2)\psi^+_{\lambda_7},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_8} &= (-a_2-a_3-1,-a_1,a_1+a_2+a_3+a_4+1;b_3/2)\psi^+_{\lambda_8},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_1} &= (-a_3-a_4-1,-a_2,-a_1;(b_4-1)/2)\psi^-_{\lambda'_1},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_2} &= (-a_1-a_2-1,a_1+a_2+a_3+1,a_4;(b_2+1)/2)\psi^-_{\lambda'_2},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_3} &= (-a_4,-a_3,-a_1-a_2-1;(b_2+1)/2)\psi^-_{\lambda'_3},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_4} &= (a_3,-a_1-a_2-a_3-1,a_1+a_2+a_3+a_4+1;(b_2+1)/2)\psi^-_{\lambda'_4},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_5} &= (a_3+a_4+1,-a_2-a_3-a_4-1,-a_1;(b_4-1)/2)\psi^-_{\lambda'_5},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_6} &= (a_1+a_2+1,a_3,a_4;(b_2+1)/2)\psi^-_{\lambda'_6},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_7} &= (-a_2,a_2+a_3+a_4+1,-a_1-a_2-a_3-a_4-1;(b_4-1)/2)\psi^-_{\lambda'_7},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_8} &= (a_1,a_2,a_3+a_4+1;(b_4-1)/2)\psi^-_{\lambda'_8}. \qquad (2.21)
\end{aligned}
$$

In case $a_1 = a_2 = a_3 = a_4 = 0$, the kernel solutions (2.21) can be grouped in terms of su(4) dimensions as follows:

$$
\mathbf{1}_{-1} \equiv \psi^+_{\lambda_1} \sim (0,0,0)_{-1}, \quad \mathbf{6}_0 \equiv \begin{cases} \psi^+_{\lambda_2} \sim (\ 0,\ 1,\ 0)_0 \\ \psi^+_{\lambda_4} \sim (\ 1,-1,\ 1)_0 \\ \psi^+_{\lambda_8} \sim (-1,\ 0,\ 1)_0 \\ \psi^+_{\lambda_3} \sim (\ 1,\ 0,-1)_0 \\ \psi^+_{\lambda_7} \sim (-1,\ 1,-1)_0 \\ \psi^+_{\lambda_5} \sim (\ 0,-1,\ 0)_0 \end{cases}, \quad \mathbf{1}_1 \equiv \psi^+_{\lambda_6} \sim (0,0,0)_1,
$$

$$
\mathbf{4}_{-1/2} \equiv \begin{cases} \psi^-_{\lambda'_1} \sim (-1,\ 0,\ 0)_{-1/2} \\ \psi^-_{\lambda'_5} \sim (\ 1,-1,\ 0)_{-1/2} \\ \psi^-_{\lambda'_7} \sim (\ 0,\ 1,-1)_{-1/2} \\ \psi^-_{\lambda'_8} \sim (\ 0,\ 0,\ 1)_{-1/2} \end{cases}, \qquad \mathbf{4}_{1/2} \equiv \begin{cases} \psi^-_{\lambda'_6} \sim (\ 1,\ 0,\ 0)_{1/2} \\ \psi^-_{\lambda'_2} \sim (-1,\ 1,\ 0)_{1/2} \\ \psi^-_{\lambda'_4} \sim (\ 0,-1,\ 1)_{1/2} \\ \psi^-_{\lambda'_3} \sim (\ 0,\ 0,-1)_{1/2} \end{cases}.
$$

Since the Dynkin labels $a_{1,2,3,4}$ are non-negative, the direct sum of the su(4) highest weights

$$
\psi^+_{\lambda_1} \oplus \psi^+_{\lambda_2} \oplus \psi^+_{\lambda_6} \oplus \psi^-_{\lambda'_8} \oplus \psi^-_{\lambda'_6}, \qquad (2.22)
$$

or in terms of its Dynkin labels,

$$
\begin{aligned}
&(a_1,a_2,a_3)_{(b_5-2)/2} \oplus (a_1,a_2+a_3+1,a_4)_{b_3/2} \oplus (a_2,a_3,a_4)_{(b_1+2)/2} \\
&\oplus (a_1,a_2,a_3+a_4+1)_{(b_4-1)/2} \oplus (a_1+a_2+1,a_3,a_4)_{(b_2+1)/2}, \qquad (2.23)
\end{aligned}
$$

forms the Euler number multiplet.

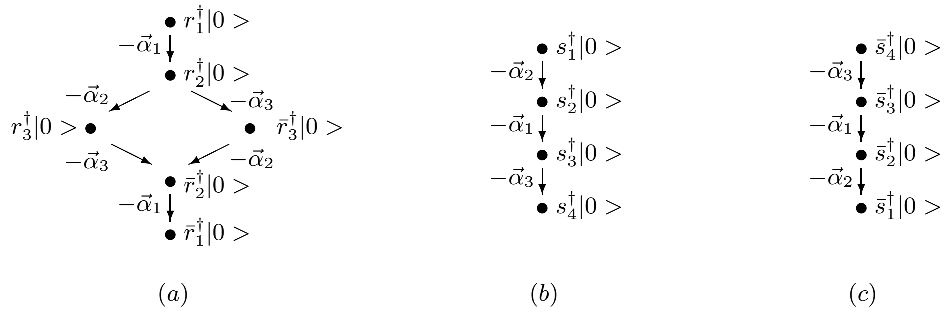

Figure 2: The so(6) weight diagrams (a) of a 6-dimensional vector (b) of a 4-dimensional co-spinor and (c) of a 4-dimensional spinor representations.

3. Kostant operator of the quotient so(6)/so(4) × so(2) and its kernel solutions

3.1 The Schwinger's oscillator realization of the so(6) Lie algebra

To construct the generators for so(6), we introduce four types of Schwinger's oscillators $r_i,\ \bar{r}_i,\ s_j,\ \bar{s}_j$, where $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, including their adjoints. Action of the raising oscillators $r_i^\dagger,\ \bar{r}_i^\dagger,\ s_i^\dagger$ and $\bar{s}_j^\dagger$ on the vacuum state in correspondence to the so(6) irreps **6**, $\mathbf{4}_c$ and $\mathbf{4}_s$ is shown in figure 2a, 2b and 2c, respectively. Although, the **6** irrep is not fundamental and can be obtained from an anti-symmetric product of two copies of either $\mathbf{4}_c$ or $\mathbf{4}_s$ irrep, it will be seen later that introducing the oscillators r_j and $\bar{r}_j$ is an easy way to determine the kernel solutions of the Kostant operator.

From the weight diagrams of so(6), all positive root generators are

$$
\begin{aligned}
T_1^+ &= r_1^\dagger r_2 + s_2^\dagger s_3 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_2^+ &= r_2^\dagger r_3 + s_1^\dagger s_2 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_3^+ &= r_2^\dagger \bar{r}_3 + s_3^\dagger s_4 + (r, \bar{r}, s \to \bar{r}, r, \bar{s})^\dagger, \\
T_4^+ &= r_1^\dagger r_3 - s_1^\dagger s_3 - (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_5^+ &= r_1^\dagger \bar{r}_3 + s_2^\dagger s_4 - (r, \bar{r}, s \to \bar{r}, r, \bar{s})^\dagger, \\
T_6^+ &= -r_1^\dagger \bar{r}_2 + s_1^\dagger s_4 + (r, \bar{r}, s \to \bar{r}, r, \bar{s})^\dagger,
\end{aligned} \tag{3.1}
$$

and all negative root generators are

$$
T_A^- = \left(T_A^+\right)^\dagger, \qquad A = 1,\ 2,\ 3,\ \ldots,\ 6. \tag{3.2}
$$

The Cartan subalgebra generators in the Dynkin basis,

$$
\vec{H} \equiv H_1 \hat{\omega}_1 + H_2 \hat{\omega}_2 + H_3 \hat{\omega}_3 = (H_1, H_2, H_3), \tag{3.3}
$$

are obtained from the following commutators:

$$
\begin{aligned}
H_1 &\equiv [T_1^+, T_1^-] = N_1^{(r)} + N_2^{(s)} - N_2^{(r)} - N_3^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_2 &\equiv [T_2^+, T_2^-] = N_2^{(r)} + N_1^{(s)} - N_3^{(r)} - N_2^{(s)} - (r, s \to \bar{r}, \bar{s}),
\end{aligned}
$$

$$H_3 \equiv [T_3^+, T_3^-] = N_2^{(r)} + N_3^{(s)} - N_3^{(\bar{r})} - N_4^{(s)} - (r, \bar{r}, s \to \bar{r}, r, \bar{s}). \tag{3.4}$$

They are related to the Cartan generators in the orthonormal basis,

$$\vec{h} \equiv h_1 \hat{e}_1 + h_2 \hat{e}_2 + h_3 \hat{e}_3 = [h_1, h_2, h_3], \tag{3.5}$$

as follows:

$$H_1 = h_1 - h_2, \;\; H_2 = h_2 - h_3, \;\; H_3 = h_2 + h_3. \tag{3.6}$$

For an so(6) irrep, its highest weight is

$$\Lambda = (r_1^\dagger)^{a_1} (s_1^\dagger)^{a_2} (\bar{s}_4^\dagger)^{a_3} |0>, \tag{3.7}$$

where $a_{1,2,3}$ are non-negative integers. Action of the so(6) Cartan generators in the Dynkin basis on it yields

$$\vec{H}\Lambda = (a_1, a_2, a_3)\Lambda, \tag{3.8}$$

and in the orthonormal basis

$$\vec{h}\Lambda = [b_1, b_2, b_3]\Lambda, \tag{3.9}$$

where

$$\begin{aligned} b_1 &= \frac{1}{2}(a_3 + a_2) + a_1, \\ b_2 &= \frac{1}{2}(a_3 + a_2), \\ b_3 &= \frac{1}{2}(a_3 - a_2). \end{aligned} \tag{3.10}$$

Inside the so(6) generators, the generators $T_{1,6}^{\pm}$ and $H_{1,6}$ form the so(4) Lie subalgebra and the generator h_3 is the generator of so(2) subalgebra. The other generators $T_{2,3,4,5}^{\pm}$ lie outside the subalgebra so(4) $\times$ so(2) and they are used to construct the Kostant operator of the quotient so(6)/so(4) $\times$ so(2).

3.2 Kernel solutions of the Kostant operator

To construct the Kostant operator of the quotient so(6)/so(4) $\times$ so(2), the gamma matrices used here are

$$\begin{aligned} \gamma_2^{\pm} &= \frac{1}{2}(\Gamma_1 \pm i\Gamma_2), \qquad \gamma_3^{\pm} = \frac{1}{2}(\Gamma_3 \pm i\Gamma_4), \\ \gamma_4^{\pm} &= \frac{1}{2}(\Gamma_5 \pm i\Gamma_6), \qquad \gamma_5^{\pm} = \frac{1}{2}(\Gamma_7 \pm i\Gamma_8). \end{aligned} \tag{3.11}$$

From the commutator of the generators of the quotient,

$$\begin{aligned} [T_2^+, T_2^-] &= h_2 - h_3, \qquad [T_3^+, T_3^-] = h_2 + h_3, \\ [T_4^+, T_4^-] &= h_1 - h_3, \qquad [T_5^+, T_5^-] = h_1 + h_3, \end{aligned} \tag{3.12}$$

the generators $T_{a=2,3,4,5}^{\pm}$ are not generated. The structure constants associated with these transformations are zero. Hence, the Kostant operator is just

$$\not{K} = \sum_{a=2}^{5} (\gamma_a^+ T_a^- + \gamma_a^- T_a^+). \tag{3.13}$$

A vector space of the Kostant operator is $\psi^{\pm}_{\Lambda} \equiv |\pm\pm\pm\pm> \otimes V_{\Lambda}$. Here, V_{Λ} is the vector space of the so(6) irrep with its highest weight Λ. For the kernel solutions

$$\not{K}\psi^{\pm}_{\lambda_i} = 0, \tag{3.14}$$

where λ_i is a weight in the vector space V_{Λ}. It is noted that the derivation of the kernel solutions $\psi^{+}_{\lambda_3}$ and $\psi^{+}_{\lambda_8}$ in this quotient is not straightforward as the one in su(5)/su(4)×u(1). At first glance, the following two equations,

$$\begin{aligned}(T_2^+ + T_3^- - T_4^- + T_5^+)\psi^{+}_{\lambda_3} &= 0,\\ (T_2^- - T_3^+ - T_4^+ - T_5^-)\psi^{+}_{\lambda_8} &= 0,\end{aligned} \tag{3.15}$$

have kernel solutions as follows:

$$\begin{aligned}\psi^{+}_{\lambda_3} &= |++-+>|0>,\\ \psi^{+}_{\lambda_8} &= |+--->|0>.\end{aligned} \tag{3.16}$$

These solutions are true only when $a_1 = a_2 = a_3 = 0$. We fix this problem by twisting their spinor states and obtain the general kernel solutions in the positive spinor space as follows:

$$\begin{aligned}\psi^{+}_{\lambda_1} &= |++++>(r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\ \psi^{+}_{\lambda_2} &= |+++->(r_1^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\ \psi^{+}_{\lambda_3} &= |+--->(r_2^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\ \psi^{+}_{\lambda_4} &= |++-->(r_3^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\ \psi^{+}_{\lambda_5} &= |+-++>(\bar{r}_1^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}|0>,\\ \psi^{+}_{\lambda_6} &= |+-+->(\bar{r}_1^\dagger)^{a_1}(s_4^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}|0>,\\ \psi^{+}_{\lambda_7} &= |+--+>(\bar{r}_3^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\ \psi^{+}_{\lambda_8} &= |++-+>(\bar{r}_2^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\end{aligned} \tag{3.17}$$

and in the negative spinor space as follows:

$$\begin{aligned}\psi^{-}_{\lambda'_1} &= |-+++>(r_2^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}|0>,\\ \psi^{-}_{\lambda'_2} &= |-++->(\bar{r}_2^\dagger)^{a_1}(s_4^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\ \psi^{-}_{\lambda'_3} &= |-+-+>(\bar{r}_1^\dagger)^{a_1}(s_4^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}|0>,\\ \psi^{-}_{\lambda'_4} &= |-+-->(\bar{r}_1^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}|0>,\\ \psi^{-}_{\lambda'_5} &= |--++>(r_2^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\ \psi^{-}_{\lambda'_6} &= |--+->(\bar{r}_2^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}|0>,\\ \psi^{-}_{\lambda'_7} &= |---+>(r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\ \psi^{-}_{\lambda'_8} &= |---->(r_1^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>.\end{aligned} \tag{3.18}$$

To get the kernel solutions in terms of so(4) × so(2), it needs to act on them by the Cartan generators, which in the Dynkin basis are

$$
\begin{aligned}
D_1 &= h_1 - h_2 + \frac{1}{2}\left(f^4_{+-1}[\gamma_4^+,\gamma_4^-] + f^5_{+-1}[\gamma_5^+,\gamma_5^-] - f^2_{+-2}[\gamma_2^+,\gamma_2^-] - f^3_{+-2}[\gamma_3^+,\gamma_3^-]\right)\\
&= H_1 + \frac{1}{2}(\mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1} + \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1} - \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3 - \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3),\\
D_2 &= h_1 + h_2 + \frac{1}{2}\left(f^4_{+-1}[\gamma_4^+,\gamma_4^-] + f^5_{+-1}[\gamma_5^+,\gamma_5^-] + f^2_{+-2}[\gamma_2^+,\gamma_2^-] + f^3_{+-2}[\gamma_3^+,\gamma_3^-]\right)\\
&= H_1 + H_2 + H_3 + \frac{1}{2}(\mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1} + \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\\
&\qquad + \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3),\\
D_3 &= \frac{1}{2}h_3 + \frac{1}{4}\left(f^2_{+-3}[\gamma_2^+,\gamma_2^-] + f^3_{+-3}[\gamma_3^+,\gamma_3^-] + f^4_{+-3}[\gamma_4^+,\gamma_4^-] + f^5_{+-3}[\gamma_5^+,\gamma_5^-]\right)\\
&= \frac{1}{2}h_3 - \frac{1}{4}(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3 - \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3 + \mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1}\\
&\qquad - \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1}).
\end{aligned}
\tag{3.19}
$$

The structure constants in (3.19) are read directly from (3.12). The generators D_1 and D_2 are the Cartan generators of so(4) and the generator D_3 is the Cartan generator of so(2). When the Cartan generators act on the kernel solutions, they give

$$
\begin{aligned}
(D_1,D_2;D_3)\psi^+_{\lambda_1} &= (a_1, a_1+a_2+a_3+2; b_3/2)\psi^+_{\lambda_1},\\
(D_1,D_2;D_3)\psi^+_{\lambda_2} &= (a_1+a_2+a_3+2, a_1; -b_3/2)\psi^+_{\lambda_2},\\
(D_1,D_2;D_3)\psi^+_{\lambda_3} &= (-a_1, a_1+a_2+a_3; b_3/2)\psi^+_{\lambda_3},\\
(D_1,D_2;D_3)\psi^+_{\lambda_4} &= (a_2, a_3; (b_1+2)/2)\psi^+_{\lambda_4},\\
(D_1,D_2;D_3)\psi^+_{\lambda_5} &= (-a_1-a_2-a_3-2, -a_1; -b_3/2)\psi^+_{\lambda_5},\\
(D_1,D_2;D_3)\psi^+_{\lambda_6} &= (-a_1, -a_1-a_2-a_3-2; b_3/2)\psi^+_{\lambda_6},\\
(D_1,D_2;D_3)\psi^+_{\lambda_7} &= (a_3, a_2; -(b_1+2)/2)\psi^+_{\lambda_7},\\
(D_1,D_2;D_3)\psi^+_{\lambda_8} &= (a_1+a_2+a_3, -a_1; -b_3/2)\psi^+_{\lambda_8},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_1} &= (-a_1-a_3-1, a_1+a_2+1; -(b_2+1)/2)\psi^-_{\lambda'_1},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_2} &= (a_1+a_3+1, -a_1-a_2-1; -(b_2+1)/2)\psi^-_{\lambda'_2},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_3} &= (-a_1-a_3-1, -a_1-a_2-1; -(b_2+1)/2)\psi^-_{\lambda'_3},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_4} &= (-a_1-a_2-1, -a_1-a_3-1; (b_2+1)/2)\psi^-_{\lambda'_4},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_5} &= (-a_1-a_2-1, a_1+a_3+1; (b_2+1)/2)\psi^-_{\lambda'_5},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_6} &= (a_1+a_2+1, -a_1-a_3-1; (b_2+1)/2)\psi^-_{\lambda'_6},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_7} &= (a_1+a_3+1, a_1+a_2+1; -(b_2+1)/2)\psi^-_{\lambda'_7},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_8} &= (a_1+a_2+1, a_1+a_3+1; (b_2+1)/2)\psi^-_{\lambda'_8}.
\end{aligned}
\tag{3.20}
$$

In case $a_1 = a_2 = a_3 = 0$, the kernel solutions (3.20) can be grouped in terms of so(4) dimensions as follows:

$$
(\mathbf{1},\mathbf{1})_1 \equiv \psi^+_{\lambda_4} \sim (\ 0,\ 0)_1,
$$

$$(\mathbf{1},\mathbf{3})_0 \equiv \begin{cases} \psi^+_{\lambda_1} \sim (\ 0,\ 2)_0 \\ \psi^+_{\lambda_8} \sim (\ 0,\ 0)_0 \\ \psi^+_{\lambda_6} \sim (\ 0,-2)_0 \end{cases} , \qquad (\mathbf{3},\mathbf{1})_0 \equiv \begin{cases} \psi^+_{\lambda_2} \sim (\ 2,\ 0)_0 \\ \psi^+_{\lambda_3} \sim (\ 0,\ 0)_0 \\ \psi^+_{\lambda_5} \sim (-2,\ 0)_0 \end{cases} ,$$

$$(\mathbf{1},\mathbf{1})_{-1} \equiv \psi^+_{\lambda_7} \sim (\ 0,\ 0)_{-1},$$

$$(\mathbf{2},\mathbf{2})_{1/2} \equiv \begin{cases} \psi^-_{\lambda'_8} \sim (\ 1,\ 1)_{1/2} \\ \psi^-_{\lambda'_6} \sim (\ 1,-1)_{1/2} \\ \psi^-_{\lambda'_5} \sim (-1,\ 1)_{1/2} \\ \psi^-_{\lambda'_4} \sim (-1,-1)_{1/2} \end{cases} , \qquad (\mathbf{2},\mathbf{2})_{-1/2} \equiv \begin{cases} \psi^-_{\lambda'_7} \sim (\ 1,\ 1)_{-1/2} \\ \psi^-_{\lambda'_2} \sim (\ 1,-1)_{-1/2} \\ \psi^-_{\lambda'_1} \sim (-1,\ 1)_{-1/2} \\ \psi^-_{\lambda'_3} \sim (-1,-1)_{-1/2} \end{cases} . \tag{3.21}$$

Since the Dynkin labels $a_{1,2,3}$ are non-negative, the direct sum of the so(4) highest weights,

$$\psi^+_{\lambda_4} \oplus \psi^+_{\lambda_1} \oplus \psi^+_{\lambda_2} \oplus \psi^+_{\lambda_7} \oplus \psi^-_{\lambda'_8} \oplus \psi^-_{\lambda'_7}, \tag{3.22}$$

or in terms of its Dynkin labels,

$$\begin{aligned}(a_2,a_3)_{(b_1+2)/2} \oplus (a_1, a_1+a_2+a_3+2)_{b_3/2} \oplus (a_1+a_2+a_3+2, a_1)_{-b_3/2} \oplus (a_3,a_2)_{-(b_1+2)/2} \\ (a_1+a_2+1, a_1+a_3+1)_{(b_2+1)/2} \oplus (a_1+a_3+1, a_1+a_2+1)_{-(b_2+1)/2},\end{aligned} \tag{3.23}$$

forms the Euler number multiplet.

4. Remarks

Kernel solutions of the Kostant operator of the 8-dimensional quotients can be easily determined by the quantum mechanical method. The Euler number multiplet obtained in terms of the diagonal subalgebra is the direct sum of the highest weights of the kernel solutions, which appear only once. The Euler number multiplets presented in this paper are exactly the same as derived by using the Weyl group elements of su(5) and so(6) that are not in their subalgebras [3]. The lowest line of the Euler number multiplet for the quotient su(5)/su(4) × u(1) is

$$1_1 \oplus 4_{1/2} \oplus 6_0 \oplus 4_{-1/2} \oplus 1_{-1}, \tag{4.1}$$

and for the quotient so(6)/so(4) × so(2)

$$(1,1)_1 \oplus (2,2)_{1/2} \oplus (3,1)_0 \oplus (1,3)_0 \oplus (2,2)_{-1/2} \oplus (1,1)_{-1}. \tag{4.2}$$

There are many possible ways to interpret these Euler number multiplets. If so(2), which is locally isomorphic to u(1), is viewed as a light-cone little group of ISO(3, 1), then they correspond to degrees of freedom of $N = 4$ Yang-Mills massless representation in 3+1 space-time. Similarly, if so(6), which is locally isomorphic to su(4), is viewed as a light-cone little group of ISO(7, 1), then they correspond to degrees of freedom of the massless representation in 7+1 space-time. Lastly, if so(6) × so(2), which is locally isomorphic to su(4) × u(1), is viewed as a subgroup of SO(6, 2), the anti-de Sitter group and the conformal group, then they correspond to the massless representations in the 6+1 and 5+1 space-time, respectively.

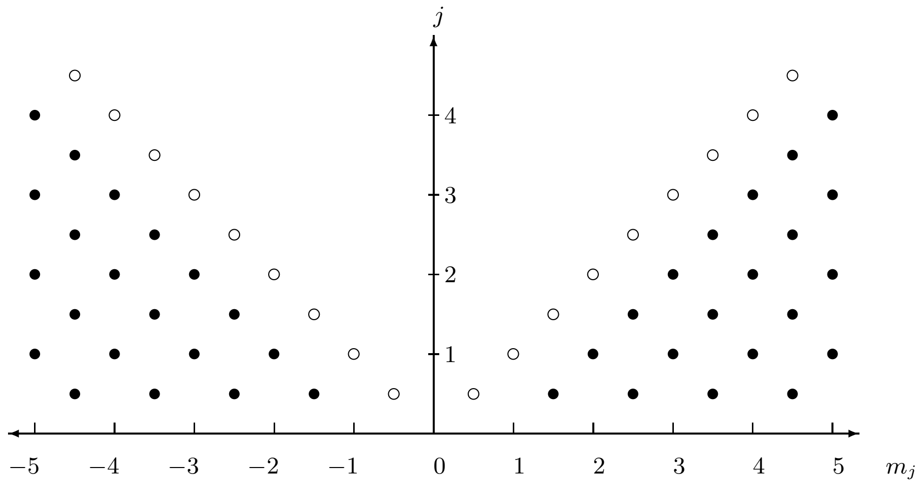

Figure 3: The so(2, 1) weight diagram associated with the discrete representations. Open and solid circles along a horizontal line are the so(2, 1) weights of a j representation. In each horizontal line, only the open circles, the lowest weight in the V_j^+ and the highest weight in the V_j^-, are the non-trivial kernel solutions of the Kostant operator of the quotient so(2, 1)/so(2).

The Kostant operator can be extended from a compact Lie algebra to a non-compact one. Methods to construct the Kostant operator are similar in both the compact and the non-compact Lie algebras. The simplest quotient of the non-compact Lie algebras is so(2, 1)/so(2). For details of the so(2, 1) generators, commutation relations and representations, see [8]. The Kostant operator,

$$\not{K} = \sigma^+ T^- + \sigma^- T^+, \tag{4.3}$$

acts on its vector space $\psi_j^\pm = |\pm > |j, m_j >$, where in each discrete representation j, $|m_j| \geq j$. Its non-trivial kernel solutions, whose corresponding states are shown as open circles in figure 3, are

$$\psi_j^+ = |+ > |j, -j >, \qquad \psi_j^- = |- > |j, j > . \tag{4.4}$$

These solutions are similar to the kernel solutions of su(2)/u(1). Another interesting non-compact Lie algebra is so(4, 2), the conformal group in the 3+1 space-time, whose spinors are twistors [9]. For the case so(4, 2)/so(4)×so(2), it is found that its lowest line of the Euler number multiplet for the discrete representation is similar to that of so(6)/so(4) × so(2).

Finally, it is just a hope that the constructions of the Kostant operators and the derivations of their kernel solutions presented here will be useful when someone wants to oxidize a low-dimensional field theory to a higher-dimensional one or to reduce a high-dimensional field theory to a lower-dimensional one [10], or even to connect the Kostant operators to the string theory [11, 12].

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